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Topological Sectors and Measures on Moduli Space in Quantum Yang-Mills on a Riemann Surface

Dana Stanley Fine
Department of Mathematics
University of Massachusetts Dartmouth
North Dartmouth, MA 02747
dfine@umassd.edu

Previous path integral treatments of Yang-Mills on a Riemann surface automatically sum over principal fiber bundles of all possible topological types in computing quantum expectations. This paper extends the path integral formulation to treat separately each topological sector. The formulation is sufficiently explicit to calculate Wilson line expectations exactly. Further, it suggests two new measures on the moduli space of flat connections, one of which proves to agree with the small-volume limit of the Yang-Mills measure. ©1996 American Institute of Physics.

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I Introduction

In Refs. [1] and [2], we use the path integral formalism to evaluate quantum expectations of Wilson lines in the Yang-Mills theory on $G = SU(N)$ product bundles over Riemann surfaces of any genus. Other approaches to these expectations include Sengupta's stochastic quantization of Ref. [3] and Blau & Thompson's use of the Nikolai map to simplify the gauge-fixed path integral in Ref. [4]. Witten, in Ref. [5], derives these expectations combinatorially and via a Hilbert-space approach using axioms of quantum field theory for any simple Lie group G . He notes that these results, by contrast with an approach based on Verlinde's formula, automatically sum over all topological types. In the more recent treatment of Ref. [6], he describes how to modify his Hilbert space approach to treat separately each topological sector. Likewise, in Refs. [7] and [8], Sengupta extends the stochastic quantization to non-simply connected G in a manner which treats separately each topological sector.

In this paper, we analyze the path integral formulation of Yang-Mills on a principal G -bundle of fixed topological type. While doing so, we augment Witten's results on the partition function by including the insertion of Wilson lines. These agree with Sengupta's recent results. Our main focus, however, is on the path integral itself. There is a recognized scarcity of field theories wherein the path integral can be evaluated (or interpreted) non-perturbatively to obtain something approaching a well-defined measure on configuration space. With the present extension of the detailed account of how to perform such an evaluation (in a manifestly gauge-invariant fashion), we hope to contribute to the eventual rigorous understanding of gauge-theoretic path integration.

As an immediate benefit, in addition to the above-mentioned extension of Witten's results, we obtain new measures on the moduli space of flat connections. For genus $g \geq 1$, the moduli space \mathcal{M} of (irreducible) flat connections is a finite-dimensional manifold. Witten uses the small-volume limit of Yang-Mills to compute the volume of \mathcal{M} , which he shows agrees with that defined by the symplectic volume form on \mathcal{M} . In Ref. [9], Forman extends this agreement to one between measures on \mathcal{M} . Our approach suggests two new measures on \mathcal{M} . We show that one of these, which was arrived at independently by King and Sengupta in Ref. [10], is equivalent to the small-volume limit of the Yang-Mills measure.

This paper is organized as follows:

Section II gives a brief summary of results for $G = SU(N)$ product bundles.

Section III describes the modifications required to keep track of individual topological sectors in treating bundles with non-trivial topology.

Section IV introduces the new measures on \mathcal{M} and compares them with the small-volume limit of the Yang-Mills measure.

II Review of the product case

Let P be a principal bundle with symmetry group G over a Riemann surface M with a base point $m \in M$. Let \mathcal{A} denote the space of connections on P , and let \mathcal{G}_m denote the group of gauge transformations which are the identity on the fiber over m . In Ref. [2], we describe $\mathcal{A}/\mathcal{G}_m$ as itself a principal bundle with affine-linear fiber over $Path^{2g} G$, the space of $2g$ -tuples of paths in G subject to the following relation on the $4g$ endpoint values $\{\alpha_i(0), \alpha_i(1), \beta_i(0), \beta_i(1)\}_{i=1}^g$:

$$\prod_{i=1}^g \alpha_i(0) \beta_i(1)^{-1} \alpha_i(1)^{-1} \beta_i(0) = \mathbf{1}. \quad (1)$$

Here, successive factors multiply from the right, and $g \geq 1$. The projection $\xi : \mathcal{A}/\mathcal{G}_m \rightarrow Path^{2g} G$ is obtained from holonomy about a certain one-parameter family of closed paths in M determined by a choice of fundamental domain and $2g$ generators $\{a_i, b_i\}_{i=1}^g$ of $\pi_1(M)$; the unusual naming of the components of the elements of $Path^{2g} G$ reflects this genesis. In the genus-0 case, the base space is simply ΩG , the space of based loops in G .

As the holonomies giving rise to ξ enter into the account of topological sectors, we review them in more detail here. Let D be a fundamental domain for M and consider a family of paths from the base point m to the boundary ∂D of D such that every point of M except m lies on exactly one such path. Call these radial paths. Let p be a point of an edge of ∂D corresponding to a generator of the fundamental group of M , and let $p^{-1} \in \partial D$ denote the point corresponding to p in the identification of the edges of ∂D . Consider, for a given connection A , the holonomy of the closed path originating at m , following the radial path to p and returning from the radial path through p^{-1} . See figure 1. Relative to a fixed point on the fiber over m , this holonomy determines an element of G which we denote by $\alpha_i(p)$ if p is in the edge corresponding to the generator a_i ($\beta_i(p)$ if the corresponding generator is b_i). As p varies along its edge, $\alpha_i(p)$ describes a path in G . The $2g$ -tuple of such paths in G described as p moves among the edges corresponding to generators is $\xi([A]) \in Path^{2g} G$. The endpoint relation reflects the fact that the radial paths to the vertices of ∂D are each traversed twice. In particular, the concatenation of paths whose holonomy is $\prod_{i=1}^g \alpha_i(0) \beta_i(1)^{-1} \alpha_i(1)^{-1} \beta_i(0)$ is a contractible path containing no area, so the holonomy is the identity in accordance with Eq. 1. The fiber of ξ is the affine-linear space $\ker P_r$ of Lie-algebra-valued one-forms which vanish in the radial direction.

In this picture, the path integral for the expectation of some function on $\mathcal{A}/\mathcal{G}_m$, in the Yang-Mills measure μ , is an iterated integral over the linear fibers and the

base $Path^{2g} G$. The integral over the fibers is Gaussian. Performing this integral yields a path integral expression for the push-down measure $\xi_*(\mu)$. The main result of Ref. [2] is that $\xi_*(\mu)$ is the product of Wiener measures on the components $\{\alpha_i, \beta_i\}$ of the elements of $Path^{2g} G$, conditioned to satisfy the endpoint relation of Eq. 1. This measure is computed by integrating products of heat kernels on G . For example, when $g = 1$, the partition function Z is given by

$$Z = \int H\left(\alpha(0)^{-1}\alpha(1); 2\rho_a\right) H\left(\beta(0)^{-1}\beta(1); 2(\rho - \rho_a)\right),$$

where the integral is over all possible values of $\alpha(0)$, $\alpha(1)$ and $\beta(0)$, and $\beta(1)$ is determined by the relation $\alpha(0)\beta(1)^{-1}\alpha(1)^{-1}\beta(0) = \mathbf{1}$. Here, H is the heat kernel, ρ is the total area of the surface, and ρ_a is the area bounded by the pair of paths whose holonomies determine $\alpha(0)$ and $\alpha(1)$. The convolution property of the heat kernel reduces this expression to

$$Z = \int H\left((\beta(0)^{-1}\alpha(1)^{-1}\beta(0)\alpha(1); 2\rho\right) d\beta(0)d\alpha(1).$$

More generally, for genus $g \geq 1$,

$$Z = \int H\left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}; 2\rho\right) dx dy. \quad (2)$$

Since

$$H(x; t) = \sum_{\mu} (\dim \mu) \chi_{\mu}(x) e^{-2c(\mu)}, \quad (3)$$

where χ_{μ} is the character of the representation μ , and c denotes the quadratic Casimir, the partition function is the sum

$$Z = \sum_{\mu} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}}.$$

All sums are over the irreducible representations of G . The induction step in reducing the integral expression of Eq. 2 to the above sum is given by a pair of integral (orthogonality) relations among the characters:

$$\begin{aligned} \int \chi_{\mu}(wxyx^{-1}y^{-1}) dy &= \frac{\chi_{\mu}(wx) \chi_{\mu}(x^{-1})}{\dim \mu}, \text{ and} \\ \int \chi_{\mu}(wx) \chi_{\mu}(x^{-1}) dx &= \frac{\chi_{\mu}(w)}{\dim \mu}. \end{aligned}$$

This formalism also treats the insertion of Wilson lines. For example, the expectation of an unknotted Wilson line, in the representation μ , given by the trace of holonomy about a non-contractible, homotopically non-trivial loop C , is

$$\langle \mathcal{W}_{\mu} \rangle = \frac{1}{Z} \int \chi_{\mu}(x_1) H\left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}; 2\rho\right) dx dy.$$

As a sum over characters, this is

$$\langle \mathcal{W}_\mu \rangle = \frac{1}{Z} \sum_\nu D_{\mu\nu} \frac{e^{-4c_\nu \rho}}{(\dim \nu)^{2g-2}},$$

where $D_{\mu\nu}$ is the Clebsch-Gordan coefficient: $\chi_\mu(x)\chi_\nu(x) = \sum D_{\mu\nu\sigma} \chi_\sigma(x)$.

To make sense of the path integral, we restricted connections in \mathcal{A} to have finite Yang-Mills action and to satisfy a continuity restriction. Without a refinement of this restriction, bundles P of different topological types are indistinguishable. Thus, although nominally we treated only product bundles, our results, when naively extended to non-simply-connected symmetry groups, correspond to a sum over all topological types.

III Separating the topological sectors

To sort out the topological sectors in the case where G is not simply-connected and $g \geq 1$, let \tilde{G} be the covering group of G . Then $G = \tilde{G}/\Gamma$, where Γ is a subgroup $\{\mathbf{1}, u_1, \dots, u_n\}$ of the finite center of the simply-connected Lie group \tilde{G} . The topological type of P can be characterized as follows: Consider holonomy by a flat connection about contractible paths in M . As elements of G , these must be the identity. However, if we lift to \tilde{G} , these holonomies, though equal to each other, can be any element u of Γ . This element defines the topological type of P .

The description in Sec. II of ξ goes through as before to yield the same endpoint relation for $Path^{2g} G$. However, if we attempt to lift from G to \tilde{G} , the holonomy of the right-hand side of the relation in Eq. 1 will be replaced by the element u of Γ labelling the topological type of P . This follows from the fact that the concatenation contains no area (so the holonomy is the same as for a flat connection) and is contractible. We have thus proven the required refinements of the main theorems of Refs. [1] and [2]:

Theorem 3.1 *On a principal fiber bundle of topological type u , $\mathcal{A}/\mathcal{G}_m$ is itself a fiber bundle with projection ξ and affine-linear fiber. The base space $Path^{2g} \tilde{G}$ consists of all $2g$ -tuples of paths in \tilde{G} , subject to the relation*

$$\prod_{i=1}^g \alpha_i(0) \beta_i(1)^{-1} \alpha_i(1)^{-1} \beta_i(0) = u. \quad (4)$$

Theorem 3.2 *The push-down measure $\xi_*(\mu)$ is the product of Wiener measures on the components of each element of $Path^{2g} \tilde{G}$, conditioned to satisfy Eq. 4.*

These allow us to calculate the partition function and the expectation of Wilson lines on a bundle of type u over a surface of genus g . For example, the calculation of the partition function for a bundle of type u on the torus ($g = 1$) begins as before:

$$Z(u) = \int H(\alpha(0)^{-1} \alpha(1); 2\rho_a) H(\beta(0)^{-1} \beta(1); 2(\rho - \rho_a)).$$

Now, however, $\beta(1)$ is determined by the relation $\alpha(0)\beta(1)^{-1}\alpha(1)^{-1}\beta(0) = u$. Thus,

$$Z(u) = \int H\left(\beta(0)^{-1}\alpha(1)^{-1}\beta(0)\alpha(1)u^{-1}; 2\rho\right) d\beta(0)d\alpha(1).$$

More generally, for genus $g \geq 1$,

$$Z(u) = \int H\left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} u^{-1}; 2\rho\right) dx dy.$$

In all these expressions the integrals are over copies of \tilde{G} and H is the heat kernel on \tilde{G} . Decomposing H as a sum of characters (of representations of \tilde{G}) according to Eq. 3,

$$Z(u) = \sum_{\mu} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}} \frac{\chi_{\mu}(u^{-1})}{(\dim \mu)}. \quad (5)$$

This agrees with Witten's results from Ref. [6][Sec. 4] except for a constant factor depending on G and g but not on ρ .

Incorporating Wilson lines given by parallel transport about the radial paths used to define ξ is as straight-forward as in the product case. For instance, the expectation of the Wilson line $\chi_{\sigma}(\alpha_i(p))$, given by the trace of holonomy about a non-contractible, homologically non-trivial loop, is

$$\begin{aligned} \langle \chi_{\sigma}(\alpha_i(p)) \rangle &= \frac{1}{Z(u)} \int \chi_{\sigma}(x) H\left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} u^{-1}; 2\rho\right) dx dy \\ &= \sum_{\mu} D_{\sigma\mu\mu} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}} \frac{\chi_{\mu}(u^{-1})}{(\dim \mu)}. \end{aligned}$$

By contrast, the expectation of a Wilson line coming from a contractible loop is

$$\begin{aligned} \langle \chi_{\sigma}(\alpha_i(p_1)^{-1} \alpha_i(p_2)) \rangle &= \frac{1}{Z(u)} \int \chi_{\sigma}(\bar{x}) H(\bar{x}; 2\rho_a) \\ &\quad \times H\left(\bar{x}^{-1} \prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} u^{-1}; 2(\rho - \rho_a)\right) d\bar{x} dx dy \\ &= \sum_{\mu\nu} D_{\sigma\mu\nu} \frac{e^{-4c_{\nu}(\rho - \rho_a)} e^{-4c_{\mu}\rho_a}}{(\dim \nu)^{2g-2}} \frac{(\dim \mu)}{(\dim \nu)} \frac{\chi_{\nu}(u^{-1})}{(\dim \nu)}, \end{aligned}$$

where ρ_a is the area enclosed by the contractible loop.

Given the ability to disentangle the topological sectors, it is an amusing exercise to compute the expected value of the topology of a random bundle. That is, let $f : \Gamma \rightarrow R$, and, viewing f as a map from topological sectors to the reals, define its Yang-Mills expectation taken over bundles of all topological types (G and g are fixed) by

$$\langle f \rangle = \frac{\sum_{u \in \Gamma} f(u) Z(u)}{\sum_{u \in \Gamma} Z(u)}. \quad (6)$$

To evaluate this expression, re-write Eq. 5 as

$$Z(u) = \sum_{\mu} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}} \lambda_{\mu}(u^{-1}),$$

where $\lambda_{\mu}(u) \equiv \frac{\chi_{\mu}(u)}{\dim \mu}$. Note that λ_{μ} is a character for the representation μ of Γ . Let $\text{Rep } \tilde{G}$ denote the set of equivalence classes of irreducible representations of \tilde{G} , and, for fixed $\alpha \in \text{Rep } \tilde{G}$, let $(\text{Rep } \tilde{G})_{\alpha}$ denote the set of representations which agree with α on Γ . Expanding $f : \Gamma \rightarrow R$ in characters as $f(u) = \sum_{\alpha \in \text{Rep } \tilde{G}} \tilde{f}_{\alpha} \lambda_{\alpha}(u)$, and letting $\alpha = 0$ denote the trivial representation of \tilde{G} , we shall prove

Corollary 3.2.1

$$\langle f \rangle = \sum_{\alpha \in \text{Rep } \tilde{G}} f_{\alpha} \frac{\sum_{\mu \in (\text{Rep } \tilde{G})_{\alpha}} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}}}{\sum_{\mu \in (\text{Rep } \tilde{G})_0} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}}}.$$

Proof: The character λ_{μ} satisfies the orthogonality relation

$$\sum_{u \in \Gamma} \lambda_{\mu}(u) \lambda_{\nu}(u^{-1}) = \begin{cases} \#\Gamma & \text{if } \lambda_{\mu}(u) = \lambda_{\nu}(u) \text{ for all } u \in \Gamma \\ 0 & \text{otherwise} \end{cases}.$$

Thus the α -th component of the numerator in Eq. 6 is

$$\sum_{u \in \Gamma} \lambda_{\alpha}(u) Z(u) = \sum_{\mu \in \text{Rep } \tilde{G}} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}} \sum_{u \in \Gamma} \lambda_{\alpha}(u) \lambda_{\mu}(u^{-1}) = \#\Gamma \sum_{\mu \in (\text{Rep } \tilde{G})_{\alpha}} \frac{e^{-4c_{\mu}\rho}}{(\dim \mu)^{2g-2}}.$$

Moreover, $1 = \lambda_0(u)$, so the denominator in Eq. 6 is the same expression with α replaced by 0. \square

Remark 3.1 Since $(\text{Rep } \tilde{G})_0 = \text{Rep } G$, the evaluation of the denominator proves the statement that naively applying the results of Refs. [1] and [2] to topologically non-trivial bundles is equivalent to summing over topologies.

Remark 3.2 Let $G = SO(3) = SU(2)/\Gamma$ for $\Gamma = \{\mathbf{1}, -\mathbf{1}\}$, and $f(\pm\mathbf{1}) = \pm 1$. Labelling the irreducible representations of $SU(2)$ by their dimensions, which span the positive integers, the odd-integer representations of $SU(2)$ are trivial on Γ , while the even-integer representations are not (that is, $\lambda_n(-\mathbf{1}) = (-1)^{n+1}$). With the conventions of

Ref. [1], the Casimir is $c(\mu) = \frac{1}{8}(n^2 - 1)$. Thus, the expected value of the topology of an $SO(3)$ -bundle is

$$\langle f \rangle = \frac{\sum_{n \text{ even}} \frac{e^{-\frac{1}{2}(n^2-1)\rho}}{n^{2g-2}}}{\sum_{n \text{ odd}} \frac{e^{-\frac{1}{2}(n^2-1)\rho}}{n^{2g-2}}}.$$

IV Measures on the moduli space of flat connections

Let \mathcal{M}_m denote the space of flat connections modulo gauge transformations. As Witten describes in Ref. [5], there is a natural symplectic form ω on \mathcal{M}_m which defines a measure $\mu_\omega = \frac{1}{\#Z(G)} \frac{\omega^n}{n!}$. Here $n = \frac{1}{2} \dim \mathcal{M}_m$. Sengupta has shown in Ref. [11] that the small-volume limit of the Yang-Mills measure μ on $\mathcal{A}/\mathcal{G}_m$ described above defines, at least in genus 0, a second measure μ_0 on \mathcal{M}_m . Witten shows these two measures agree on the total volume of \mathcal{M}_m , and in Ref. [9] Forman shows that, in fact, $\mu_\omega = \mu_0$.

The picture of $\mathcal{A}/\mathcal{G}_m$ as a bundle over $Path^{2g} \tilde{G}$ suggests a new measure on \mathcal{M}_m . First note that two points of \mathcal{M}_m cannot lie in the same fiber, as there is no 1-form $\tau \in \ker P_r$ for which $D_A \tau = 0$. Thus, the restriction of ξ to \mathcal{M}_m , henceforth denoted $\xi|_{\mathcal{M}_m}$, is invertible. Moreover, $\xi(\mathcal{M}_m) = \{\vec{\gamma} \in Path^{2g} \tilde{G} : \gamma_i \text{ is constant}\}$. (This is another way of saying that \mathcal{M}_m may be viewed as the representations of $\pi_1(M)$ on \tilde{G} .) In short, $\xi|_{\mathcal{M}_m}$ provides an isomorphism between \mathcal{M}_m and \tilde{G}_z^{2g} , the space of constant $2g$ -tuples in $Path^{2g} \tilde{G}$. In Theorem 4.2 of Ref. [2], we exhibited a global section $\sigma : Path^{2g} G \rightarrow \mathcal{A}/\mathcal{G}_m$. Here we note the restriction of σ to \tilde{G}_z^{2g} is $\xi|_{\mathcal{M}_m}^{-1}$. This is an immediate consequence of the fact that, in a given fiber, the connection A representing a point in the image of σ is determined up to gauge transformation by the condition that $\langle F_A, D_A \tau \rangle$ vanishes for all $\tau \in \ker P_r$.

To define a measure on \mathcal{M}_m , use the Haar measure on G to define a measure on \tilde{G}_z^{2g} and then use σ to push this measure forward to \mathcal{M}_m . In detail, the Haar measure on G defines a measure μ_H on \tilde{G}_z^{2g} which is the product of Haar measures on the components of \tilde{G}^{2g} , conditioned to satisfy the endpoint relation. That is,

$$\mu_H(x_1, y_1, \dots, x_g, y_g) = \delta \left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1} z^{-1} \right) dx dy,$$

where δ denotes the Dirac delta distribution massed at the identity of \tilde{G} .

Under the isomorphism $\xi|_{\mathcal{M}}$, μ_H defines a measure μ_ξ on \mathcal{M}_m . As a measure on functions on \mathcal{M}_m , μ_ξ is given as:

$$\int_{\mathcal{M}_m} f \mu_\xi = \int_{\tilde{G}^{2g}} f \circ \sigma(\vec{x}, \vec{y}) \mu_H. \quad (7)$$

Note that the right-hand side is an obvious measure to define on \mathcal{M}_m viewed as representations of $\pi_1(M)$. This measure may be normalized to define $\langle f \rangle_{\mu_\xi}$, the expectation of a function on \mathcal{M}_m .

In Ref. [10], King and Sengupta arrive at the measure μ_ξ by a construction analogous to the construction of ξ reviewed in Sec. II. (They, however, restrict from the outset to connections representing elements of \mathcal{M}_m .) They then argue directly that $\mu_\xi = \mu_\omega$. Here, by contrast, we will show that $\mu_\xi = \mu_0$. More precisely,

Theorem 4.1 *Up to normalization, $\mu_\xi = \mu_0$ on functions which are analytic along the fibers of $\mathcal{A}/\mathcal{G}_m$.*

Proof: We shall show $\lim_{\rho \rightarrow 0} \langle f \rangle_\mu = \langle f|_{\mathcal{M}_m} \rangle_{\mu_\xi}$. Writing out the path integral expression for $\langle f \rangle_\mu$ and performing the Gaussian integral over the fibers yields

$$\langle f \rangle_\mu = \frac{1}{Z} \int_{Path^{2g} \tilde{G}} \hat{f}(\vec{\gamma}) \xi_*(\mu)(\vec{\gamma})$$

where \hat{f} is obtained from $f(\sigma(\vec{\gamma}) + \tau)$ by performing the Gaussian integral over $\tau \in \ker P_\tau$. Since $\xi_*(\mu)$ is the product of Wiener measures, and the latter are determined by their behavior on cylinder sets, we may assume, without loss of generality, that $\hat{f}(\vec{\gamma})$ depends on α_i and β_i evaluated at points p_{ij} and q_{ik} , respectively, where $j = 1, 2, \dots, m_i$ and $k = 1, 2, \dots, n_i$ (and $i = 1, 2, \dots, g$). Then, as in the examples of Sec. II,

$$\begin{aligned} \langle f \rangle_\mu &= \left\langle \hat{f}(\alpha_1(p_{11}), \alpha_1(p_{12}), \dots, \beta_g(q_{gn_g})) \right\rangle_{\xi_*(\mu)} = \\ &= \frac{1}{Z} \int \hat{f}(x_{11}, x_{12}, \dots, y_{gn_g}) \prod_{j=1}^{m_i+1} H(x_{i(j-1)}^{-1} x_{ij}; \Delta t_{ij}) \\ &\times \prod_{k=1}^{n_i+1} H(y_{i(k-1)}^{-1} y_{ik}; \Delta s_{ik}) \delta \left(\prod_{i=1}^g x_{i0} y_{i(n_i+1)}^{-1} x_{i(m_i+1)}^{-1} y_{i0} z^{-1} \right) \prod_{i,j,k} dx_{ij} dy_{ik}, \quad (8) \end{aligned}$$

where $\Delta t_{i0} = 0$, Δt_{ij} is twice the area between the radial loops through p_{ij} and $p_{i(j-1)}$ and Δs_{ik} is defined similarly. As ρ approaches 0, so do Δt_{ij} and Δs_{ik} . However, as Δt approaches 0, $H(x; \Delta t)$ becomes a delta function massed at x . Thus, if we may take the limit prior to integrating,

$$\begin{aligned} \lim_{\rho \rightarrow 0} \langle f \rangle_\mu &= \\ &= \frac{1}{Z} \int \hat{f}(x_{11}, x_{12}, \dots, y_{gn_g}) \prod_{j=1}^{m_i+1} \delta(x_{i(j-1)}^{-1} x_{ij}) \\ &\times \prod_{k=1}^{n_i+1} \delta(y_{i(k-1)}^{-1} y_{ik}) \delta \left(\prod_{i=1}^g x_{i0} y_{i(n_i+1)}^{-1} x_{i(m_i+1)}^{-1} y_{i0} z^{-1} \right) \prod_{i,j,k} dx_{ij} dy_{ik}. \end{aligned}$$

Performing the integrations over x_{ij} and y_{ik} for $j, k \neq 0$ changes all the x_{ij} 's and y_{ik} 's to x_{i0} 's and y_{i0} 's, respectively, leaving

$$\lim_{\rho \rightarrow 0} \langle f \rangle_\mu = \frac{1}{Z} \int_{\tilde{G}^{2g}} \hat{f}(x_{10}, x_{10}, \dots, y_{g0}) \delta \left(\prod_{i=1}^g x_{i0} y_{i0}^{-1} x_{i0}^{-1} y_{i0} z^{-1} \right) \prod_i dx_{i0} dy_{i0}. \quad (9)$$

The right-hand side is exactly $\frac{1}{Z} \int \hat{f}|_{\xi(\mathcal{M}_m)} \mu_H$. The assumption we made about interchanging limit and integration is

$$\lim_{t \rightarrow 0} \int f(x) H(y^{-1}x; t) dx = \int f(x) \delta(y^{-1}x) dx.$$

That each side is equal to $f(y)$ follows from the definition of the heat kernel (and the continuity in t of solutions of the heat equation) on the left and the definition of the distribution δ on the right. We thus have

$$\lim_{\rho \rightarrow 0} \langle f \rangle_\mu = \frac{1}{Z} \int \hat{f}|_{\xi(\mathcal{M}_m)} \mu_H.$$

According to Eq. 7, we must now show

$$\lim_{\rho \rightarrow 0} \hat{f} = f|_{\mathcal{M}_m} \circ \sigma. \quad (10)$$

First, note that if f is constant along the fibers, that is, if $f(\sigma(\vec{\gamma}) + \tau) = f \circ \sigma$, then, for any ρ , $\hat{f} = f \circ \sigma$. More generally, if $f(\sigma(\vec{\gamma}) + \tau)$ is an n th order polynomial in τ , then \hat{f} is an n th order polynomial in ρ , whose constant term is $f \circ \sigma$. This follows from standard manipulations of Gaussian integrals and the fact that, in two dimensions, the area ρ plays the role of the coupling constant. Thus, for f polynomial, or, more generally, analytic, in τ , Eq. 10 holds (with the convergence being uniform); hence, up to normalization, $\mu_\xi = \mu_0$ on analytic functions. \square

Remark 4.1 The restriction to analytic functions of the fiber is not terribly severe. In most field theory, polynomials are sufficient. Moreover, the freedom in choosing the fundamental domain is sufficient to ensure that a large class of Wilson lines may be realized as functions which are *constant* on each fiber.

Theorem 4.1 provides a new proof of Forman's generalization of Sengupta's result:

Corollary 4.1.1 *The small-volume limit of μ is supported on \mathcal{M}_m .*

Observe that only the restriction of f to \mathcal{M}_m enters into the above calculation of $\langle f \rangle_{\mu_0}$. Specifically, let χ_R be the indicator function of a measurable set $R \subset \mathcal{A}/\mathcal{G}_m$

for which $R \cap \mathcal{M}_m = \emptyset$, and let χ_R^{smooth} be a smooth, non-negative function which is 1 on R and has support in the complement of \mathcal{M}_m . Now, let $\{\chi_R^n\}$ be a sequence of analytic functions converging uniformly to χ_R^{smooth} . Then, by the theorem, $\langle \chi_R^n \rangle_{\mu_0} = \langle \chi_R^n|_{\mathcal{M}_m} \rangle_{\mu_\xi}$. By the construction of the sequence, there is some N such that χ_R^n vanishes on \mathcal{M}_m for all $n > N$. Hence, $\langle \chi_R^{smooth} \rangle_{\mu_0} = 0$ and thus $\langle \chi_R \rangle_{\mu_0} = 0$.

Remark 4.2 We have assumed the existence of the sequence $\{\chi_R^n\}$. If $\mathcal{A}/\mathcal{G}_m$ were a finite-dimensional manifold, the Stone-Weierstrass Theorem would ensure the existence of such a sequence, but at present the above “proof” of the corollary is a heuristic argument.

Comparing the arguments to \hat{f} in Eq. 8 and Eq. 9 shows that one effect of going to the small-volume limit is to project from $Path^{2g} \tilde{G}$ to \tilde{G}_z^{2g} by evaluating each component path at $t = 0$. Denote this evaluation map by $e : Path^{2g} \tilde{G} \rightarrow \tilde{G}_z^{2g}$. This projection suggests another measure μ_e on \mathcal{M}_m which is the push-forward of μ by the projection from $\mathcal{A}/\mathcal{G}_m$ to \mathcal{M}_m given by $\sigma \circ e \circ \xi$. That is,

$$\mu_e = \sigma_* e_* \xi_*(\mu).$$

As σ is an isomorphism and $\xi_*(\mu)$ is the product of Wiener measures described in Sec. III, the only new feature is the effect of pushing forward by e . This means integrating using the Wiener measures with fixed left end-points. The derivation of Eq. 2, with minor modifications to work in a fixed topological sector and to integrate only over right end-points, leads to

$$\langle f \rangle_{\mu_e} = \frac{1}{Z} \int f \circ \sigma(x_1, \dots, y_g) H \left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}; 2\rho \right) dx_i dy_i.$$

Comparing this with $\langle f \rangle_{\mu_\xi}$ (c.f. Eq. 7), it is clear that, up to normalization,

$$\mu_e = \frac{H \left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}; 2\rho \right)}{\delta \left(\prod_{i=1}^g x_i y_i x_i^{-1} y_i^{-1}; 2\rho \right)} \mu_\xi.$$

The measure μ_e is thus a new measure on \mathcal{M}_m which agrees with μ_ξ only in the limit as ρ approaches 0.

V Conclusion

We have extended the path integral formulation of Yang-Mills on Riemann surfaces to treat each topological sector separately. The result is in agreement with Witten’s approach and is sufficiently explicit to compute quantum expectations of a large class

of Wilson lines. It also provides a new measure on \mathcal{M}_m , the moduli space of flat connections.

There are many routes to defining measures on \mathcal{M}_m . The symplectic form ω on \mathcal{M}_m and the small-volume limit of μ define the measures μ_ω and μ_0 , respectively. The view of $\mathcal{A}/\mathcal{G}_m$ as a bundle over $Path^{2g} G$ suggests the measures μ_ξ and μ_e . However, this apparent profusion of measures on \mathcal{M}_m is in fact a pair of measures. Combining Theorem 4.1 with Forman's result:

$$\mu_\xi = \mu_0 = \mu_\omega.$$

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